BROWN UNIVERSITY Providence, Rhode Island • 02912

DIVISION OF ENGINEERING

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Semi-Annual Status Report

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Variational Methods for Solving Heat Conduction Problems

Skeleton summary

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Report

Item 1. Use of a transcendental approximation in transient conduction with a non-linear boundary condition; short-time and long-time asymptotic solutions.

A report "Use of a transcendental approximation in transient conduction analysis" by P. D. Richardson and W. W., Smith, which describes this work, was submitted to NASA in July, 1965. (Note that there is a typographical error in the bottom line of p. 11 of this report, which should read

$$\psi = 1.157\ \tau^{1/2}; \qquad \chi = 2.59\ \tau^{1/2}$$

and in Table II, where the first number in the third column should read 1.1053 instead of 1.053.) The solutions obtained appear to be of much higher accuracy than those found previously.

Item 2. Transient conduction with a non-linear boundary condition; middle-time solutions.

The solutions obtained for the problem examined in Item 1 are asymptotic for large and small times. It appears that these are approached rapidly from "middle" time, but it is desirable to examine this directly. Steps were taken to program the middle-time solution for the TR-10 analog computer of the Division of Engineering. Because of the limitations of the machine, it is necessary to run the problem with ψ and χ^2 as the dependent variables (ψ and χ are the dimensionless surface temperature and the penetration depth respectively, as defined in Item 1), and to use the machine-mounted function generator to provide $\frac{d}{d\tau}(\chi^2)$ as a boundary condition (it changes from one constant value to another in going from one asymptotic solution to another). This function generator has to be adaptively operated such that the solutions generated satisfy another boundary condition of the problem. The running of the program has been temporarily suspended while the computer is in use for scheduled teaching purposes.

Item 3. Transient conduction in a passive probe, used to measure adaptive response of human skin heat transfer under sudden change of surface temperature.

Real conduction problems are excellent sources of examples for testing techniques. Measurements made in a passive, solid probe used to examine the response of limited areas of human skin to sudden change of environmental temperature (such as can be caused in practice by gripping a cold or hot control handle) provide data of this sort. Different methods of solution are being compared, and some results are expected shortly from a program being run on the IBM 7090 computer at Imperial College, London (this at no cost to the grant). Besides providing a testing ground, this item should eventually yield quantitative information about transient conduction in human skin.

Item 4. Combined transient conduction and radiation

Amongst the conduction problems of present practical interest is conduction in a structure subjected to radiation from external sources and itself. For example, some pre-launch estimates of satellite temperature have proved to be in significant error. Thermal transients can arise in tumbling. The transient thermal behavior of systems in which radiation is an important transport mechanism have not been discussed very much, except when the radiation forms a simple boundary condition, e.g. (1,2). An attempt is being made here to use variational methods to investigate transient temperature distributions in a configuration where self-radiation is important. The configuration is of an infinite set of straight, parallel, opaque plates, arranged like a venetian blind; however, instead of tilting the blind to decrease the normal distance between adjacent plates, the plate distance can be varied, i.e. the aspect ratio of the gap can be varied arbitrarily. One side of the set is exposed

1. T. J. Lardner, AIAA Jl. 1, 196-206 (1963)
2. R. D. Zerkle & J. E. Sutherland, ASME Jl. Heat Transfer 87, 117-133 (1965)

to radiation from a constant temperature source; the other side is similarly exposed up to time $t = 0$, after which the source of radiation on this side is completely removed. The simple case where there is no conduction involved and the final steady state temperature distribution is sought, subject to the assumption that the fourth-power of temperature varies linearly from one edge of the plates to the other, corresponds with that discussed by E. M. Sparrow (3), and the same results were obtained. The final steady state solution with longitudinal heat conduction was obtained in a similar way except that the governing equation was linearized; for plate gap/length ratios which are small and with reasonable orders of plate thickness and thermal conductivity, the contribution of longitudinal conduction is of the order of a few per cent, as would be expected on physical grounds. The transient case, subject to similar linearization, with uniform lumped heat capacity and with longitudinal conduction excluded, has been examined; a feature of the solution is that at short times and with the plate gap/width ratio less than about 0.5 there exists a portion of each plate near the high-temperature end at which the plate surface temperature remains essentially at its pre-transient temperature. The local surface temperature variations with time at various positions along the plates are being applied as boundary conditions for transient conduction transverse across the plates. As a consequence of this the effective lumped heat capacity is determined as a function of time, and the solution of the variational principle revised to account for this improved estimate of the lumped heat capacity.

Item 5. Approximation of repeated integrals of the error function

As a consequence of the work for the report mentioned in Item 1, it was realized that the profile function used in the report can provide excellent approximations for repeated integrals of the error function. The repeated integrals of the error function are useful, for example, in the solution of transient conduction in a semi-infinite solid where the prescribed surface temperature can be expressed as a polynomial in $(\text{time})^{1/2}$. Use is hampered on the one hand by the limitations of the tables available, and on the other by continual loss of significant digits if an attempt is made to generate values on a digital computer with repeated application of a recurrence relation. The n-th normalized repeated integral, $i^n\text{erfc}(y)/i^n\text{erfc}(0)$, can be approximated well by the profile function with appropriate values of a_n and b_n. A study showed that determination of a_n and b_n from a least-squares fitting procedure was computationally horrible, and good values can be obtained by matching the gradients at $y = 0$ and the integrals from zero to infinity. Typical values are $a_1 = 1.093$; $a_{10} = 3.026$, with

$$b_n \cdot \exp a_n = 2\Gamma(\tfrac{n}{2}+1)/\Gamma(\tfrac{n-1}{2}+1) \quad .$$

As a consequence of this, an economical representation of the repeated integrals can be made in a computer, requiring only input of a_n and b_n and use of the standard exponential subroutine.

The approximate inter-relation of the repeated integrals of the error function and particular profile functions appears to have further uses, but these have not been explored.

3. E. M. Sparrow, Advances in Heat Transfer, vol. II, p. 431

These results are incidental to the main line of investigation, but deserve mention.

Personnel

The following persons have contributed to the work mentioned in this status report:

D. Cygan, Graduate Student — NASA Trainee
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